MAKING GROUP TOPOLOGIES WITH, AND WITHOUT, CONVERGENT SEQUENCES

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Abstract. (1) Every infinite, Abelian compact (Hausdorff) group K admits $2^{|K|}$-many dense, non-Haar-measurable subgroups of cardinality $|K|$. When K is nonmetrizable, these may be chosen to be pseudocompact.

(2) Every infinite Abelian group G admits a family $\mathcal{A}$ of $2^{2^{|G|}}$-many pairwise nonhomeomorphic totally bounded group topologies such that no nontrivial sequence in G converges in any of the topologies $\mathcal{T} \in \mathcal{A}$. (For some G one may arrange $w(G, \mathcal{T}) < 2^{|G|}$ for some $\mathcal{T} \in \mathcal{A}$.)

(3) Every infinite Abelian group G admits a family $\mathcal{B}$ of $2^{2^{|G|}}$-many pairwise nonhomeomorphic totally bounded group topologies, with $w(G, \mathcal{T}) = 2^{|G|}$ for all $\mathcal{T} \in \mathcal{B}$, such that some fixed faithfully indexed sequence in G converges to 0_G in each $\mathcal{T} \in \mathcal{B}$.

1. Introduction

Historical Background 1.1. Not long after M. H. Stone and E. Čech associated with each Tychonoff space X its maximal compactification $\beta(X)$ (the so-called Stone-Čech compactification), it was noted, denoting by ω the countably infinite discrete space, that $\beta(\omega)$ contains no nontrivial convergent sequence. This observation stimulated Efimov [13] to pose in 1969 a question which in its full generality remains unsolved today: Does every compact, Hausdorff space contain either a copy of $\beta(\omega)$ or a nontrivial convergent sequence? (In models of $\diamondsuit$ the answer is negative [15]. See [32] for several additional relevant references.) The present paper is concerned with topological groups. In that context, a correct and natural companion to Efimov's question is this: Given a class $\mathcal{C}$ of topological groups, does every group in $\mathcal{C}$ contain a nontrivial convergent sequence? There is an extensive

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literature on questions of this form. Here are some samples of both positive and negative results.

Positive (a) According to a result to which Šapirovskiĭ, Gerlits and Efimov have contributed (see [33] for historical details and for an "elementary" proof), every infinite compact group K contains topologically a copy of the generalized Cantor set $\{0,1\}^{w(K)}$, hence contains a convergent sequence; (b) Assuming GCH, Malykhin and Shapiro [27] showed that every totally bounded group G with $w(G) < (w(G))^\omega$ contains a nontrivial convergent sequence; (c) Raczkowski [29], [30] and others [1] have shown that for every suitably fast-growing sequence $x_n \in \mathbb{Z}$ there is a totally bounded group topology on $\mathbb{Z}$ with respect to which $x_n \to 0$.

Negative. Glicksberg [17] showed that when a locally compact Abelian group $(G,\mathcal{T})$ is given its associated Bohr topology (that is, the weak topology induced on G by $\widehat{(G,\mathcal{T})}$), no new compact sets are created; in particular, as shown earlier by Leptin [26], the topology induced on a (discrete) Abelian group by $\mathrm{Hom}(G,\mathbb{T})$ has no infinite compact subsets, in particular has no nontrivial convergent sequence; (b) there are infinite pseudocompact topological groups containing no nontrivial convergent sequence [34].

Perhaps the most celebrated unsolved question in this area of mathematics is this: Does there exist in ZFC a countably compact topological group with no nontrivial convergent sequence? (Many examples are known in augmented axiom systems. See for example the constructions of van Douwen [11], of Hart and van Mill [20], and of Tomita [38], [39], and see also [10] for a characterization, in a forcing model of ZFC + CH with $2^{\mathfrak{c}}$ "arbitrarily large", of those Abelian groups which admit a hereditarily separable pseudocompact (alternatively, countably compact) group topology with no infinite compact subsets.)

Outline 1.2. The present work achieves results which in a certain direction may legitimately be called optimal: We show that every infinite Abelian group G admits the maximal number, namely $2^{2^{|G|}}$, of totally bounded group topologies with no nontrivial convergent sequence (Theorem 4.1); and also, the same number of totally bounded group topologies in each of which some nontrivial sequence (fixed, and chosen in advance) does converge (Theorem 5.5). An elementary cardinality argument shows that the various topological groups $(G,\mathcal{T})$ may be chosen to be pairwise nonhomeomorphic as topological spaces.

Notation 1.3. The symbols κ and α denote infinite cardinals; ω is the least infinite cardinal, and $\mathfrak{c} := 2^\omega$. For S a set we write $[S]^\kappa := \{A \subseteq S : |A| = \kappa\}$; the symbols $[S]^{<\kappa}$ and $[S]^{\leq\kappa}$ are defined analogously. $\mathbb{Z}$ and $\mathbb{R}$ denote respectively the group of integers and the group of real numbers, often with their usual (metrizable) topologies, and $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. General groups G are written multiplicatively, with identity $1 = 1_G$, but groups G known or hypothesized to be Abelian are written additively, with identity $0 = 0_G$. We identify each finite cyclic group with its copy in $\mathbb{T}$; in particular

for $0 < n < \omega$ we write $\mathbb{Z}(n) := \{\frac{k}{n} : 0 \leq k < n\} \subseteq \mathbb{T}$. We denote by $\mathbb{P}$ the set of primes.

We work exclusively with Tychonoff spaces, i.e., with completely regular, Hausdorff spaces. A topological group which is a Hausdorff space is necessarily a Tychonoff space [22](8.4). A topological group $(G, \mathcal{T})$ is said to be *totally bounded* if for $\emptyset \neq U \in \mathcal{T}$ there is $F \in [G]^{<\omega}$ such that $G = FU$. It is a theorem of Weil [40] that a topological group is totally bounded if and only if G embeds densely into a compact group; this latter group, unique in an obvious sense, is called the *Weil completion* of G and is denoted $\overline{G}$.

Given a topological group $G = (G, \mathcal{T})$, the symbol $\widehat{G} = \widehat{(G, \mathcal{T})}$ denotes the set of continuous homomorphisms from G to the (compact) group $\mathbb{T}$. A group G with its discrete topology is written G_d, so $\widehat{G_d} = \mathrm{Hom}(G, \mathbb{T})$; this is a closed subgroup of the compact group $\mathbb{T}^G$. It is easily seen, as in [7](1.9), that for a subgroup H of $\mathrm{Hom}(G, \mathbb{T})$ these conditions are equivalent: (a) H separates points of G; (b) H is dense in the compact group $\mathrm{Hom}(G, \mathbb{T})$.

For G Abelian we denote by $\mathcal{S}(G)$ the set of point-separating subgroups of $\mathrm{Hom}(G, \mathbb{T})$, and by $\mathfrak{t}(G)$ the set of (Hausdorff) totally bounded group topologies on G. Theorem 1.5 below describes, for each infinite Abelian group G, a useful order-preserving bijection between $\mathcal{S}(G)$ and $\mathfrak{t}(G)$.

The rank, the torsion-free rank, and (for $p \in \mathbb{P}$) the p-rank of an Abelian group G are denoted $r(G)$, $r_0(G)$, and $r_p(G)$, respectively. We have $r(G) = r_0(G) + \sum_{p \in \mathbb{P}} r_p(G)$, and when $|G| > \omega$ we have $|G| = r(G)$ (cf. [16](§16) or [22](Appendix A)). Following those sources we write

$$G_p := \bigcup_{k<\omega} \{x \in G : p^k \cdot x = 0\}.$$

We write $X =_h Y$ if X and Y are homeomorphic topological spaces, and we write $G \simeq H$ if G and H are isomorphic groups; it is important to note that $X =_h Y$ conveys no information about the algebraic structure of X or Y (if any), and $G \simeq H$ conveys no information about the topological structure of G or H (if any).

We assume familiarity on the reader's part with the essentials of Haar measure $\lambda = \lambda_K$ on a locally compact group K. The set of Borel sets, and the set of λ-measurable sets, are denoted $\mathcal{B}(K)$ and $\mathcal{M}(K)$, respecively. Of course $\mathcal{B}(K) \subseteq \mathcal{M}(K)$. Our convention is that Haar measure is *complete* in the sense that if $S \in \mathcal{M}(K)$ with $\lambda(S) = 0$, then each $A \subseteq S$ satisfies $A \in \mathcal{M}(K)$ (with $\lambda(A) = 0$). We assume for simplicity that if K is compact then λ is *normalized* in the sense that $\lambda(K) = 1$.

The following three theorems are essential to our argument. Therem 1.4(a) is due to Steinhaus [35] when $G = \mathbb{R}$ and to Weil [41](p. 50) in the general case; see Stromberg [36] for a pleasing, efficient proof. Parts (b) and (c) are the immediate consequences which we use here frequently.

Theorem 1.4. [35], [41]. *Let K be a locally compact group and let S be a λ-measurable subset of K with $\lambda(S) > 0$. Then*

(a) *the difference set* SS^{-1} *contains a neighborhood of* 1_K;
(b) *if* S *is a subgroup of* K *then* S *is open and closed in* K*; and*
(c) *if* S *is a dense subgroup of* K *then* $S = K$. $\square$

Theorem 1.5. [7]. *Let* G *be an Abelian group.*

(a) *For every* $H \in \mathcal{S}(G)$*, the topology* $\mathcal{T}_H$ *induced on* G *by* H *is a (Hausdorff) totally bounded group topology such that* $w(G, \mathcal{T}_H) = |H|$*;*
(b) *if* $(G, \mathcal{T})$ *is totally bounded then* $\mathcal{T} = \mathcal{T}_H$ *with* $H := \widehat{(G, \mathcal{T})} \in \mathcal{S}(G)$. $\square$

It is clear with G as in Theorem 1.5 that distinct $H_0, H_1 \in \mathcal{S}(G)$ induce distinct topologies $\mathcal{T}_{H_0}, \mathcal{T}_{H_1} \in \mathfrak{t}(G)$—indeed $\widehat{(G, \mathcal{T}_{H_0})} = H_0 \neq H_1 = \widehat{(G, \mathcal{T}_{H_1})}$; thus the bijection $\mathfrak{t}(G) \leftrightarrow \mathcal{S}(G)$ given by $\mathcal{T}_H \leftrightarrow H$ is indeed order-preserving.

Theorem 1.6. [9]. *Let* G *be an infinite Abelian group with* $K := \mathrm{Hom}(G, \mathbb{T}) = \widehat{G_d}$*, let* $(x_n)_n$ *be a faithfully indexed sequence in* G*, and let*

$$A := \{h \in K : h(x_n) \to 0\}.$$

Then $A \in \mathcal{M}(K)$*, and* $\lambda(A) = 0$.

Proof. [Outline]. According to the duality theorem of Pontrjagin [28] and van Kampen [25], the map $G_d \twoheadrightarrow \widehat{\widehat{G_d}} = \widehat{\mathrm{Hom}(G, \mathbb{T})}$ given by $x \to \widehat{x}$ (with $\widehat{x}(h) = h(x)$ for $x \in G$, $h \in K$) is a bijection. Writing

$$A_{n,m} := \{h \in K : |\widehat{x_n}(h) - 1| < \frac{1}{m}\},$$

the relation $A = \bigcap_{m<\omega} \bigcup_{N \geq m} \bigcap_{n \geq N} A_{n,m}$ expresses A as a $G_{\delta\sigma\delta}$-subset of the compact group K, so $A \in \mathcal{B}(K) \subseteq \mathcal{M}(K)$. If G is torsion-free, a condition equivalent to the condition that $\mathrm{Hom}(G, \mathbb{T})$ is connected (cf. [22](24.25)), then a reference to Theorem 1.4 completes the proof: the condition $\lambda(A) > 0$ would imply $A = \mathrm{Hom}(G, \mathbb{T})$, so that $x_n \to 0$ in the Bohr topology of G_d, contrary to the theorem of Leptin and Glicksberg cited above. We refer the reader to [9] for the proof (for general Abelian G) that $\lambda(A) = 0$. $\square$

In what follows we will frequently invoke this simple algebraic fact.

Theorem 1.7. *Let* K *be an Abelian group, let* H *be a subgroup of* K *of index* $\alpha > \omega$*, and let*

$$\mathcal{H} := \{S : H \subseteq S \subseteq K,\ S \text{ is a proper subgroup of } K, |S| = |K|\}.$$

Then $|\mathcal{H}| = 2^\alpha$.

Proof. The inequality $\leq$ is obvious. We have $|K/H| = r(K/H) = \alpha > \omega$, so algebraically $K/H \supseteq \bigoplus_{\xi<\alpha} C_\xi$ with each C_ξ cyclic. Let $\phi : K \twoheadrightarrow K/H$ be the canonical homomorphism, and for $A \in [\alpha]^\alpha \backslash \{\alpha\}$ set $H_A := \phi^{-1}(\bigoplus_{\xi \in A} C_\xi)$. The map $[\alpha]^\alpha \backslash \{\alpha\} \to \mathcal{H}$ given by $A \to H_A$ is an injection, so $|\mathcal{H}| \geq 2^\alpha = 2^{|K/H|}$, as asserted. $\square$

Theorem 1.6 explains our interest in the existence of (many) point-separating nonmeasurable subgroups of compact Abelian groups K (of the form $K = \mathrm{Hom}(G, \mathbb{T})$): each such $H \in \mathcal{S}(G)$ will induce on G a totally bounded group topology without nontrivial convergent sequences. In order to show that such K admit $2^{|K|}$-many such subgroups, we find it convenient to treat separately the metrizable case (that is, $w(K) = \omega$) and the nonmetrizable case ($w(K) > \omega$). We do this in Sections 2 and 3 respectively.

2. Many Nonmeasurable Subgroups: The Metrizable Case

For simplicity, and because it suffices for our applications, we take the groups K and M in Lemma 2.4 and Theorem 2.2 to be compact; the reader may notice that this hypothesis can be significantly relaxed. Indeed both groups are Abelian and M is metrizable in our applications, but since those hypotheses save no labor we omit them for now.

Lemma 2.1. *Let K and M be compact groups with Haar measures λ and μ resectively, let $\phi : K \twoheadrightarrow M$ be a continuous surjective homomorphism, and define $\Phi : \mathcal{B}(M) \to \mathcal{B}(K)$ by $\Phi(E) := \phi^{-1}[E]$. Then the map $m := \lambda \circ \Phi : \mathcal{B}(M) \to [0, 1]$ satisfies $m = \mu|\mathcal{B}(M)$.*

Proof. According to [22](15.8), and using the numbering system there, it is enough to show that

- (iv) $m(C) < \infty$ for compact $C \in \mathcal{B}(M)$; (v) $m(U) > 0$ for some open $U \in \mathcal{B}(M)$;
- (vi) $m(a + F) = m(F)$ for all $a \in M$, $F \in \mathcal{B}(M)$; and
- (vii) $m(U) = \sup\{m(F) : F \subseteq U, F \text{ is compact}\}$ for open $U \subseteq M$, and $m(E) = \inf\{m(U) : E \subseteq U, U \text{ is open}\}$ for $E \in \mathcal{B}(M)$.

The verifications are routine and will not be reproduced here. In addition to [22], the reader seeking hints might consult [18](63C and 64H), or [18](52G and 52H). □

Theorem 2.2. *Let K and M be compact groups with Haar measures λ and μ respectively, and let $\phi : K \twoheadrightarrow M$ be a continuous, surjective homomorphism. If D is a dense, non-μ-measurable subgroup of M, then $H := \phi^{-1}(D)$ is a dense, non-λ-measurable subgroup of K.*

Proof. ϕ is an open map [22](5.29), so H is dense in K. Suppose now that $H \in \mathcal{M}(K)$, so that either $\lambda(H) > 0$ or $\lambda(H) = 0$. If $\lambda(H) > 0$ then $H = K$ by Theorem 1.4(c) so $D = \phi[K] = M$, a contradiction. If ($H \in \mathcal{M}(K)$ and) $\lambda(H) = 0$ then since λ is (inner-) regular there is a sequence K_n ($n < \omega$) of compact subsets of $K\backslash H$ such that $\lambda(\bigcup_n K_n) = \lambda(K\backslash H) = 1$. We write $M_n := \phi[K_n]$ and $\widetilde{K_n} := \phi^{-1}(M_n)$. Then $K_n \subseteq \widetilde{K_n} \subseteq K\backslash H$ and from Lemma 2.1 we have

$$\mu(\bigcup_n M_n) = \lambda(\bigcup_n \widetilde{K_n}) \geq \lambda(\bigcup_n K_n) = 1,$$

so $\mu(\bigcup_n M_n) = 1$ and hence $D \in \mathcal{M}(M)$ with $\mu(D) = 0$, a contradiction. □

Our goal is to show that every (infinite) compact Abelian metrizable group contains a dense, nonmeasurable subgroup of index $\mathfrak{c}$. We treat some special cases first. In what follows we denote the torsion subgroup of an Abelian group K by $t(K)$, for $s \in K$ and $0 \neq n \in \mathbb{Z}$ we write $[\frac{s}{n}] = \{x \in K : nx = s\}$, and for a subgroup S of K we set

$$\mathrm{div}(S) := \bigcup\{[\frac{s}{n}] : s \in S, 0 \neq n \in \mathbb{Z}\}.$$

When $[\frac{s}{n}] \neq \emptyset$ we choose $s_n \in [\frac{s}{n}]$, and we write $\Lambda(S) := \{s_n : s \in S, 0 \neq n \in \mathbb{Z}\} \cup \{0\}$. Then $|\Lambda(S)| \leq |S| \cdot \aleph_0$, and $\mathrm{div}(S) = \Lambda(S) + t(K)$.

Lemma 2.3. *Let M be an Abelian group such that $|M| = \kappa > \omega$ and let $S \in [M]^{<\kappa}$, $E \in [M]^{\kappa}$ with S a subgroup. If either*

(i) *$t(M) < \kappa$, or*

(ii) *there is $p \in \mathbb{P}$ such that $p \cdot M = \{0\}$,*

then there is $x \in E$ such that $\langle x \rangle \cap S = \{0\}$. In case (i)*, x may be chosen in $M \backslash t(M)$.*

Proof. (i) From $\mathrm{div}(S) = \Lambda(S) + t(M)$ follows $|\mathrm{div}(S)| < \kappa$, and any $x \in E \backslash \mathrm{div}(S) \subseteq E \backslash t(M)$ is as required.

(ii) Since $M \simeq \bigoplus_\kappa \mathbb{Z}(p) = \bigoplus_{\xi<\kappa} \mathbb{Z}(p)_\xi$, there is $A \in [\kappa]^{<\kappa}$ such that $S \subseteq \bigoplus_{\xi \in A} \mathbb{Z}(p)_\xi$. Any $x \in E$ such that $0 \neq x \notin \bigoplus_{\xi \in A} \mathbb{Z}(p)_\xi$ is as required. □

Theorem 2.4. *Let M be an infinite, compact, metrizable, Abelian group such that either*

(i) *$|t(M)| < \mathfrak{c}$ or*

(ii) *there is $p \in \mathbb{P}$ such that $p \cdot M = \{0\}$.*

Then M admits a dense, nonmeasurable subgroup D such that $|M/D| = \mathfrak{c}$.

In case (i) *one may arrange $D \simeq \bigoplus_\mathfrak{c} \mathbb{Z}$, in case* (ii) *one may arrange $D \simeq \bigoplus_\mathfrak{c} \mathbb{Z}(p)$.*

Proof. Let $\{F_\xi : \xi < \mathfrak{c}\}$ be an enumeration of all uncountable, closed subsets of M, and define $E_\xi := (F_\xi \backslash t(M)) \backslash \{0\}$ in case (i), $E_\xi := F_\xi \backslash \{0\}$ in case (ii). It is a theorem of Cantor [2](page 488) that each $|F_\xi| = \mathfrak{c}$ (see [21](VIII §9 II) or [14](4.5.5(b)) for more modern treatments); hence each $|E_\xi| = \mathfrak{c}$. There is $x_0 \in E_0$, and by Lemma 2.3 there is $y_0 \in E_0$ such that $\langle x_0 \rangle \cap \langle y_0 \rangle = \{0\}$.

Now let $\xi < \mathfrak{c}$, suppose that x_η, y_η have been chosen for all $\eta < \xi$, and apply Lemma 2.3 twice to choose $x_\xi, y_\xi \in E_\xi$ such that

$$\langle \{x_\xi\} \rangle \cap \langle \{x_\eta : \eta < \xi\} \cup \{y_\eta : \eta < \xi\} \rangle = \{0\}, \text{ and}$$
$$\langle \{y_\xi\} \rangle \cap \langle \{x_\eta : \eta \leq \xi\} \cup \{y_\eta : \eta < \xi\} \rangle = \{0\}.$$

Thus x_ξ, y_ξ are defined for all $\xi < \mathfrak{c}$. We define $D := \langle \{x_\xi : \xi < \mathfrak{c}\} \rangle$. Clearly $D = \bigoplus_{\xi<\mathfrak{c}} \mathbb{Z}$ in case (i), and $D = \bigoplus_{\xi<\mathfrak{c}} \mathbb{Z}(p)_\xi$ in case (ii) since $|D| = \mathfrak{c}$ and $p \cdot D = \{0\}$. For $\xi < \eta < \mathfrak{c}$ we have $y_\eta + D \neq y_\xi + D$, so $\mathfrak{c} \geq |K/D| \geq \mathfrak{c}$.

For nonempty open $U \subseteq K$ there is by the regularity of λ a (necessarily uncountable) compact set $F = F_\xi \subseteq U$ such that $\lambda(F_\xi) > 0$. Then $x_\xi \in E_\xi \cap D \subseteq F_\xi \cap D$. Thus D is dense in K. If $D \in \mathcal{M}(K)$ with $\lambda(D) > 0$

then $D = K$ by Theorem 1.4(c), contrary to the relation $|K/D| = \mathfrak{c}$. If $D \in \mathcal{M}(K)$ with $\lambda(D) = 0$ then $\lambda(K \backslash D) = 1$ and there is $F_\xi \subseteq K \backslash D$ such that $\lambda(F_\xi) > 0$; then $y_\xi \in E_\xi \cap D \subseteq D \backslash D = \emptyset$, a contradiction. $\square$

Corollary 2.5. *Let M be a (compact, Abelian, metrizable) group of one of these types.*

(i) $M = \mathbb{T}$;
(ii) $M = \Delta_p$ $(p \in \mathbb{P})$, *the group of p-adic integers;*
(iii) $M = \prod_{k<\omega} \mathbb{Z}(p_k)$, $p_k \in \mathbb{P}$, $(p_k)_k$ *faithfully indexed;*
(iv) $M = (\mathbb{Z}(p))^\omega$ $(p \in \mathbb{P})$.

Then M admits a dense, nonmeasurable subgroup D such that $|M/D| = \mathfrak{c}$.

Proof. Surely $|t(\mathbb{T})| = \omega$, and $t(\prod_{k<\omega} \mathbb{Z}(p_k))$ is the countable group $\bigoplus_{k<\omega} \mathbb{Z}(p_k)$. If $0 \neq h \in \Delta_p = \mathrm{Hom}(\mathbb{Z}(p^\infty), \mathbb{T})$, then $h[\mathbb{Z}(p^\infty)] \simeq \mathbb{Z}(p^\infty)/\ker(h) \simeq \mathbb{Z}(p^\infty)$ since $|\ker(h)| < \omega$, so $h[\mathbb{Z}(p^\infty)]$ is not of bounded order; thus $t(\Delta_p) = \{0\}$. It follows for M as in (i), (ii) and (iii) that $|t(M)| = 1 < \mathfrak{c}$ or $|t(M)| = \omega < \mathfrak{c}$, so Theorem 2.4(i) applies. For M as in (iv) surely $p \cdot M = \{0\}$, so Theorem 2.4(ii) applies. $\square$

Theorem 2.6. *Let K be an infinite, compact, Abelian metrizable group. Then ($|K| = \mathfrak{c}$ and) K has a dense, nonmeasurable subgroup H such that $|H| = \mathfrak{c}$ and $|K/H| = \mathfrak{c}$.*

Proof. The (discrete) dual group $G = \widehat{K}$ satisfies $|G| = w(K) = \omega$. As with any countably infinite Abelian group, G must satisfy (at least) one of these conditions: (i) $r_0(G) > 0$: (ii) $|G_p| = \omega$ with $r_p(G) < \omega$ for some $p \in \mathbb{P}$; (iii) $0 < r_p(G) < \omega$ for infinitely many $p \in \mathbb{P}$; (iv) $r_p(G) = \omega$ for some $p \in \mathbb{P}$. According as (i), (ii), (iii) or (iv) holds we have, respectively, $G \supseteq \mathbb{Z}$, $G \supseteq \mathbb{Z}(p^\infty)$, $G \supseteq \bigoplus_{k<\omega} \mathbb{Z}(p_k)$, or $G \supseteq \bigoplus_\omega \mathbb{Z}(p)$, so taking adjoints we have a continuous surjection ϕ from K onto a group M of the form $\widehat{\mathbb{Z}} = \mathbb{T}$, $\widehat{\mathbb{Z}(p^\infty)} = \Delta_p$, $\widehat{\bigoplus_{k<\omega} \mathbb{Z}(p_k)} = \prod_{k<\omega} \mathbb{Z}(p_k)$, or $\widehat{\bigoplus_\omega \mathbb{Z}(p)} = (\mathbb{Z}(p))^\omega$. According to Corollary 2.5 the group M has a dense, nonmeasurable subgroup D such that $|M/D| = \mathfrak{c}$, and then by Theorem 2.2 with $H := \phi^{-1}(D)$ the group H is dense and nonmeasurable in K. If $a, b \in K$ with $a + H = b + H$ then $\phi(a) + D = \phi(b) + D$, so $\mathfrak{c} = |K| \geq |K/H| \geq |M/D| = \mathfrak{c}$. $\square$

Theorem 2.7. *Let K be an infinite, compact, Abelian, metrizable group. Then K admits a family of $2^{|K|}$-many dense, nonmeasurable subgroups, each of cardinality $\mathfrak{c}$.*

Proof. Let H be as given in Theorem 2.6 and let

$$\mathcal{H} := \{S : H \subseteq S \subseteq K,\ S \text{ is a proper subgroup of } K\}.$$

Then $|\mathcal{H}| = 2^{\mathfrak{c}}$ by Theorem 1.7. Theorem 1.4(c) shows for $S \in \mathcal{H}$ that $S \in \mathcal{M}(K)$ with $\lambda(S) > 0$ is impossible, and if $S \in \mathcal{H}$ with $\lambda(S) = 0$ then $\lambda(H) = 0$, a contradiction. $\square$

Remark 2.8. For our application in Theorem 4.1 below we do not require that $|K/S| = \mathfrak{c}$, but in fact that condition does hold for $2^{\mathfrak{c}}$-many $S \in \mathcal{H}$.

3. Many Nonmeasurable Subgroups: The Nonmetrizable Case

We turn now to the case $w(K) = \kappa > \omega$. Again, our goal is to show that such a compact Abelian group K contains $2^{|K|}$-many dense, nonmeasurable subgroups of cardinality $|K|$. By a well-known structure theorem (see Theorem 3.2 below), such a group K admits a continuous, surjective homomorphism onto a group M of the form $M = \prod_{\xi<\kappa} M_\xi$ with each M_ξ a (compact) subgroup of $\mathbb{T}$, and according to Theorem 2.2 it suffices to show that such M has a family of $2^{|M|}$-many ($= 2^{|K|}$-many) such subgroups. We find it convenient to show a bit more, namely that M, and hence also K, admits a family of $2^{|M|}$-many subgroups each of which is G_δ-dense in M. In the transition, we will invoke the following lemma.

Lemma 3.1. *A proper, G_δ-dense subgroup H of a compact group K is nonmeasurable.*

Proof. As usual, using Theorem 1.4(c), $H \in \mathcal{M}(K)$ with $\lambda(H) > 0$ is impossible; it suffices then to show that $\lambda(H) = 0$ is also impossible. The following argument is from [24] and [23], as exposed by Halmos [18]. If $\lambda(K\backslash H) = 1 > 0$ there are a compact set C and a Baire set F of K such that $F \subseteq C \subseteq K\backslash H$ and $\lambda(F) = \lambda(C) > 0$ ([18](64H and p. 230)). As with any nonempty Baire set, F has the form $F = XB$ for a suitably chosen compact Baire subgroup B of K and $X \subseteq K$ ([18](64E)). Since every compact Baire set is a G_δ-set ([18](51D)), each $x \in X$ has xB a G_δ-set. From $xB \cap H = \emptyset$ it follows that H is not G_δ-dense in K, a contradiction. $\square$

Now for compact Abelian K with $w(K) = \kappa > \omega$ we write $G = \widehat{K}$, so that $|G| = w(K) = \kappa$, and we denote the torsion-free rank and (for $p \in \mathbb{P}$) the p-rank of G by $\kappa_0 = r_0(G)$ and $\kappa_p = r_p(G)$, respectively. Since $w(K) = |G| > \omega$ we have $|G| = \kappa = \kappa_0 + \sum_{p\in\mathbb{P}} \kappa_p$ (with perhaps $\kappa_i = 0$ for certain $i \in \mathbb{P} \cup \{0\}$), and algebraically

$$G \supseteq \bigoplus_{\kappa_0} \mathbb{Z} \oplus \bigoplus_{p\in\mathbb{P}} \bigoplus_{\kappa_p} \mathbb{Z}(p). \tag{$*$}$$

Theorem 3.2. *Let K be a compact, Abelian group such that $w(K) = \kappa > \omega$. Then K has a family of $2^{|K|}$-many G_δ-dense subgroups of cardinality $|K|$.*

Proof. The map ψ adjoint to the inclusion map in $(*)$ is a continuous, surjective homomorphism:

$$\psi : K = \widehat{G} \twoheadrightarrow \mathbb{T}^{\kappa_0} \times \prod_{p\in\mathbb{P}} (\mathbb{Z}(p))^{\kappa_p}$$

(with, again, perhaps $\kappa_i = 0$ for certain $i \in \mathbb{P} \cup \{0\}$). If $\kappa_0 = \kappa$ or some $\kappa_p = \kappa$ $(p \in \mathbb{P})$—as is necessarily the case in $\mathrm{cf}(\kappa) > \omega$—then a suitable projection map π furnishes a continuous, surjective homomorphism $\phi = \pi \circ \psi : K \twoheadrightarrow M := F^\kappa$ with $F = \mathbb{T}$ or $F = \mathbb{Z}(p)$. That such a compact group M admits a faithfully indexed family $\{D_\xi : \xi < 2^\kappa\}$ of G_δ-dense subgroups

is known (see [6](4.4)); according to [4](2.2) the groups $\phi^{-1}(D_\xi)$ ($\xi < 2^\kappa$) are then G_δ-dense in K. We assume now that $\kappa_0 < \kappa$ and each $\kappa_p < \kappa$ ($p \in \mathbb{P}$), so that necessarily $\mathrm{cf}(\kappa) = \omega$; we show that the "ultrafilter technique" of [4] and [6] can be adapted to cover this case also.

The set of uniform ultrafilters over an (infinite, discrete) set A is here denoted $\mathfrak{u}(A)$. It is well known that $|\mathfrak{u}(A)| = 2^{2^{|A|}}$. (See [3] for a proof and for other relevant combinatorial facts.)

Let $(p_n)_n$ be a faithfully indexed sequence in $\mathbb{P}$ such that $\omega < \kappa_{p_n} < \kappa$ and $\sup_{n<\omega} \kappa_{p_n} = \kappa$, and choose pairwise disjoint sets A_n such that $|A_n| = \kappa_{p_n}$; then with $A := \bigcup_{n<\omega} A_n$ we have $|A| = \kappa$. We write $\phi = \pi \circ \psi : K \twoheadrightarrow M = \prod_{n<\omega} \mathbb{Z}(p_n)^{A_n}$, we view A as a discrete space and we view each $f \in M$ as a function of the form $f = \bigcup_{n<\omega} f_n$ with $f_n : A_n \to \mathbb{Z}(p_n) \subseteq \mathbb{T}$. The Stone-Čech extension of f, mapping $\beta(A)$ into $\mathbb{T}$, is denoted $\overline{f}$. Now for each $q \in \mathfrak{u}(A) \subseteq \beta(A)$ we define $h_q : M \to \mathbb{T}$ by $h_q(f) = \overline{f}(q) \in \mathbb{T}$. For $f_0, f_1 \in M$ we have $\overline{f_0 + f_1}|A = f_0 + f_1 = \overline{f_0}|A + \overline{f_1}|A$, so $\overline{f_0 + f_1} = \overline{f_0} + \overline{f_1}$; thus $h_q \in \mathrm{Hom}(M, \mathbb{T})$ and $\mathrm{graph}(h_q)$ is a subgroup of $M \times \mathbb{T}$.

We claim that $\mathrm{graph}(h_q)$ is G_δ-dense in $M \times \mathbb{T}$. (It will follow in particular then that $M_q := \ker(h_q)$ is a G_δ-dense subgroup of M.)

Let $C = \{a_k\}_k$ be a faithfully indexed subset of A, say with $a_k \in A_{n_k}$, and let $v_k \in \mathbb{Z}(p_{n_k})$; further, let $w \in \mathbb{T}$. Then with $V := \{f \in M : f(a_k) = v_k\}$ and $W := \{w\}$, the set $V \times W$ is a nonempty G_δ-subset of $M \times \mathbb{T}$; since each nonempty G_δ-subset of $M \times \mathbb{T}$ contains a set of the form $V \times W$, to prove the claim it suffices to show that for each $q \in \mathfrak{u}(A)$ there is $f \in M$ such that $f(a_k) = v_k$ (that is, $f \in V$) and also $\overline{f}(q) = w$ (that is, $h_q(f) \in W$). Since $\bigcup_{n<\omega} \mathbb{Z}(p_n)$ is dense in $\mathbb{T}$ there is $w_n \in \mathbb{Z}(p_n)$ such that $w_n \to w$. We define $g \in M$ by $g_n \equiv w_n$ on A_n, and we define $f \in M$ by

$$f(a) = \begin{cases} g(a) & \text{if } a \in A \backslash C, \\ w_{n_k} & \text{if } a = a_k \in C. \end{cases}$$

Each neighborhood U of q in $\beta(A)$ meets infinitely many of the sets A_n, so $w_n \in g[U]$ for infinitely many $n < \omega$ and hence $\overline{g}(q) = w$. Clearly $f \in V$, and since $f \equiv g$ on $A \backslash C \in q$ we have $h_q(f) = \overline{f}(q) = \overline{g}(q) = w$. The claim is proved.

Since $|\mathfrak{u}(A)| = 2^{2^\kappa}$, to see that M has $2^{|M|}$-many G_δ-dense subgroups it suffices to show that if $q_0 \neq q_1$ with $q_i \in \mathfrak{u}(A)$ then $M_{q_0} = \ker(h_{q_0}) \neq \ker(h_{q_1}) = M_{q_1}$. Choose $F \subseteq A$ such that $F \in q_0$ and $A \backslash F \in q_1$, fix (for some $n < \omega$) $t \in \mathbb{Z}(p_n) \cap (\frac{1}{4}, \frac{3}{4}) \subseteq \mathbb{T}$ and define $f \in M$ by

$$f(a) = \begin{cases} 0 & \text{if } a \in F, \\ t & \text{if } a \in A \setminus F. \end{cases}$$

Then $h_{q_0}(f) = \overline{f}(q_0) = 0$ and $h_{q_1}(f) = \overline{f}(q_1) = t \neq 0$, so $f \in \ker(h_{q_0}) = M_{q_0}$ while $f \notin \ker(h_{q_1}) = M_{q_1}$.

Again as in [4](2.2), each $\phi^{-1}(M_q)$ ($q \in \mathfrak{u}(A)$) is a (proper) G_δ-dense subgroup of K.

It remains to show that the G_δ-dense subgroups of K constructed above are of cardinality $|K|$. If $|K| = 2^\kappa = \mathfrak{c}$ (as can occur for certain $\kappa > \omega$ in some models of set theory), this is clear since a G_δ-dense subgroup of a compact group is pseudocompact [8] and hence, if infinite, has cardinality at least $\mathfrak{c}$ [12], [5]. If $|K| = |M| = 2^\kappa > \mathfrak{c}$ then from $h_q[M] \simeq M/M_q$ and $|M/M_q| \leq |\mathbb{T}| = \mathfrak{c}$ we infer $|M_q| = |M|$ and hence $|K| \geq |\phi^{-1}(M_q)| \geq |M_q| = |M| = |K|$. The same argument applies to the G_δ-dense subgroups D_ξ of F^κ given in [6](4.4) which appear in the first paragraph of this proof, for each of those is the kernel of a homomorphism from F^κ onto $F \subseteq \mathbb{T}$. $\square$

We note in passing that in general not every G_δ-dense subgroup D of a compact, nonmetrizable group K satisfies $|D| = |K|$. See in this connection Remark 6.4 below.

Corollary 3.3 is now immediate from Lemma 3.1 and Theorem 3.2; and Theorem 3.4, which is (1) of our Abstract, is the conjuction of Theorem 2.7 and Corollary 3.3.

Corollary 3.3. *Let K be a compact, Abelian group such that $w(K) > \omega$. Then K admits a family of $2^{|K|}$-many dense, nonmeasurable subgroups, each of cardinality $|K|$.*

Theorem 3.4. *Every infinite, Abelian compact (Hausdorff) group K admits $2^{|K|}$-many dense, non-Haar-measurable subgroups of cardinality $|K|$. When K is nonmetrizable, these may be chosen to be pseudocompact.*

Remark 3.5. It is known [37] that $\mathbb{R}$ contains a dense non-measurable subgroup of both cardinality and index $\mathfrak{c}$. Arguing as in Theorem 2.7, we see then that $\mathbb{R}$ has $2^{\mathfrak{c}}$-many dense non-measurable subgroups of both cardinality and index $\mathfrak{c}$. We say as usual that a *compactly generated* group is a topological group generated, in the algebraic sense, by a compact subset. By [22] (9.8) for every (Hausdorff) locally compact Abelian compactly generated group G there are non-negative integers m and n and a compact Abelian group K such that G is of the form $\mathbb{R}^m \times K \times \mathbb{Z}^n$. If G is not discrete, then either $m > 0$ or K is not discrete, so $w(G) = \omega + w(K)$. It follows that a non-discrete (Hausdorff) locally compact Abelian compactly generated group G has $2^{|G|}$-many dense non-Haar-measurable subgroups of both cardinality and index $|G|$. More generally, let G be an arbitrary non-discrete (Hausdorff) locally compact Abelian group, and let H be an open compactly generated subgroup of G. It follows that G has $2^{|H|}$-many dense non-Haar-measurable subgroups $\{G_\xi : \xi < 2^{|H|}\}$, each of cardinality $|G|$, and such that each $H_\xi := G_\xi \cap H$ has $|H/H_\xi| = |H|$. We leave the details to the reader.

4. Group Topologies Without Convergent Sequences

Here we pull together the threads of Sections 2 and 3.

Theorem 4.1. *Every infinite Abelian group G admits a family $\mathcal{A}$ of totally bounded group topologies, with $|\mathcal{A}| = 2^{2^{|G|}}$, such that no nontrivial sequence in G converges in any of the topologies in $\mathcal{A}$. One may arrange in addition that*

(i) *$w(G,\mathcal{T}) = 2^{|G|}$ for each $\mathcal{T} \in \mathcal{A}$, and*

(ii) *for distinct $\mathcal{T}_0$, $\mathcal{T}_1 \in \mathcal{A}$ the spaces $(G,\mathcal{T}_0)$ and $(G,\mathcal{T}_1)$ are not homeomorphic.*

Proof. By Theorem 2.7 when $|G| = \omega$, and by Corollary 3.3 when $|G| > \omega$, the compact group $K := \mathrm{Hom}(G,\mathbb{T}) = \widehat{G_d}$ admits a family $\mathcal{H}$ of dense, nonmeasurable subgroups such that $|\mathcal{H}| = 2^{2^{|G|}}$ and $|H| = |K|$ for each $H \in \mathcal{H}$. According to Theorems 1.5 and 1.6 the family $\mathcal{A} := \{\mathcal{T}_H : H \in \mathcal{H}\}$ satisfies all requirements except (perhaps) (ii). A homeomorphism between two of the spaces $(G,\mathcal{T}_0)$, $(G,\mathcal{T}_1)$ with $\mathcal{T}_i \in \mathcal{A}$ is realized by a permutation of G, and there are just $2^{|G|}$-many such functions, so for each $\mathcal{T} \in \mathcal{A}$ there are at most $2^{|G|}$-many $\mathcal{T}' \in \mathcal{A}$ such that $(G,\mathcal{T}) =_h (G,\mathcal{T}')$. Statement (ii) then follows (with $\mathcal{A}$ replaced if necessary by a suitably chosen subfamily of cardinality $2^{|K|} = 2^{2^{|G|}}$). □

Remark 4.2. The case $G = \mathbb{Z}$ of Theorem 4.1 is not new. See in this connection [29] and [30], which were the motivation for much of the present paper.

5. Topologies With Convergent Sequences

We turn now to the complementary or opposing problem, that of finding on an arbitrary infinite Abelian group G the maximal number (that is, $2^{2^{|G|}}$-many) totally bounded group topologies in which some nontrivial sequence converges. Again, this result was first achieved for $G = \mathbb{Z}$ in [29], [30]: there, $2^{\mathfrak{c}}$ many topologies in $\mathfrak{t}(\mathbb{Z})$ are constructed in which an arbitrary sequence x_n, fixed in advance and satisfying $x_{n+1}/x_n \geq n+1$, converges to 0. The condition $x_{n+1}/x_n \geq n+1$ of [29] and [30] was relaxed in [1] to $x_{n+1}/x_n \to \infty$.

To handle the case of general Abelian G, our strategy is to show first that certain "basic" countable groups accept $2^{\mathfrak{c}}$ many such topologies. We begin with technical results concerning groups of the form $\bigoplus_{k<\omega} \mathbb{Z}(p_k^{r_k})$ and of the form $\mathbb{Z}(p^\infty)$.

We remark for emphasis that in Theorem 5.1 the given sequence $(p_k)_k$ in $\mathbb{P}$ is not necessarily faithfully indexed. Indeed the case $p_k = p \in \mathbb{P}$ (a constant sequence) is not excluded. For $x \in A = \bigoplus_{k<\omega} \mathbb{Z}(p_k^{r_k})$ we write $x = (x(k))_{k<\omega}$.

Theorem 5.1. *Let $p_k \in \mathbb{P}$ and $A = \bigoplus_{k<\omega} \mathbb{Z}(p_k^{r_k})$ with $0 < r_k < \omega$, and let $(x_n)_{n<\omega}$ be a faithfully indexed sequence in A such that*

(i) *there is $S \in [\omega]^\omega$ such that $x_n(k) = 0$ for all $n < \omega$, $k \in S$, and*

(ii) *$|\{n < \omega : x_n(k) \neq 0\}| < \omega$ for all $k < \omega$.*

Let $\{A_\xi : \xi < \mathfrak{c}\}$ *enumerate* $\mathcal{P}(S) \cup [\omega]^{<\omega}$*, and for* $\xi < \mathfrak{c}$ *define* $h_\xi \in \mathrm{Hom}(A, \mathbb{T})$ *by* $h_\xi(x) = \sum_{k\in A_\xi} x(k)$*. Then*

(a) *the set* $\{h_\xi : \xi < \mathfrak{c}\}$ *is faithfully indexed;*
(b) *the set* $\{h_\xi : \xi < \mathfrak{c}\}$ *separates points of* A*; and*
(c) $h_\xi(x_n) \to 0$ *for each* $\xi < \mathfrak{c}$*.*

Proof. If $\xi, \xi' < \mathfrak{c}$ with $\xi \neq \xi'$, say $k \in A_\xi \backslash A_{\xi'}$, then any $x \in G$ such that $0 \neq x(k) \in \mathbb{Z}(p_k^{r_k})$ and $x(m) = 0$ for $k \neq m < \omega$ satisfies

$$h_\xi(x) = x(k) \neq 0 = h_{\xi'}(x).$$

(b) Let $x, x' \in G$ with, say, $x(k) \neq x'(k)$, and let $\{k\} = A_\xi$ $(\xi < \mathfrak{c})$. Then

$$h_\xi(x) = x(k) \neq x'(k) = h_\xi(x').$$

(c) If $A_\xi \in \mathcal{P}(S)$ then $h_\xi(x_n) = 0$ for all n by (i), so $h_\xi(x_n) \to 0$. If $A_\xi \in [\omega]^{<\omega}$ then by (ii) there is $N < \omega$ such that $h_\xi(x_n) = 0$ for all $n > N$, so again $h_\xi(x_n) \to 0$. $\square$

Next, following [22] and [16], we identify the elements of the compact group $\Delta_p = \mathrm{Hom}(\mathbb{Z}(p^\infty), \mathbb{T})$ with those sequences $h = (h(k))_{k<\omega}$ of integers such that $0 \leq h(k) < p - 1$ for all $k < \omega$. For $\frac{a}{p^n} \in \mathbb{Z}(p^\infty)$ (with $0 \leq a < p^n - 1$) we have

$$h(\frac{a}{p^n}) = a \cdot h(\frac{1}{p^n}) = a \cdot \sum_{k=0}^{n-1} \frac{h(k)}{p^{n-k}} = \frac{a}{p^n} \cdot \sum_{k=0}^{n-1} h(k) \cdot p^k \pmod 1.$$

In what follows we write $\mathrm{Fac} := \{n! : n < \omega\}$.

Theorem 5.2. *Let* $(a_n)_{n<\omega}$ *be a sequence of integers such that* $0 \leq a_n < p - 1$ *for all* $n < \omega$*, and let* $x_n = \frac{a_n}{p^{n!}} \in \mathbb{Z}(p^\infty)$*. Let* $\{A_\xi : \xi < \mathfrak{c}\}$ *enumerate* $\mathcal{P}(\mathrm{Fac}) \cup [\omega]^{<\omega}$*, and for* $\xi < \mathfrak{c}$ *define* $h_\xi = (h_\xi(k))_{k<\omega} \in \Delta_p$ *by*

$$h_\xi(k) = \begin{cases} 1 & \text{if } k \in A_\xi, \\ 0 & \text{otherwise.} \end{cases}$$

Then

(a) *the set* $\{h_\xi : \xi < \mathfrak{c}\}$ *is faithfully indexed;*
(b) *the set* $\{h_\xi : \xi < \mathfrak{c}\}$ *separates points of* A*; and*
(c) $h_\xi(x_n) \to 0$ *for each* $\xi < \mathfrak{c}$*.*

Proof. The proofs of (a) and (b) closely parallel their analogues in Theorem 5.1. We prove (c). If $A_\xi \in [\omega]^{<\omega}$ there is $N < \omega$ such that

$$h_\xi(x_n) = \frac{a_n}{p^{n!}} \cdot \sum_{k=0}^{n!-1} h_\xi(k) \cdot p^k \leq \frac{p-1}{p^{n!}} \cdot \sum_{k=0}^{N} p^k \quad \text{for all } n > N,$$

so $h_\xi(x_n) \to 0$. If $A_\xi \in \mathcal{P}(\mathrm{Fac})$ then

$$\begin{aligned} h_\xi(x_n) &= \frac{a_n}{p^{n!}} \cdot \sum_{k=0}^{n!-1} h_\xi(k) \cdot p^k \\ &= \frac{a_n}{p^{n!}} \cdot \sum_{k \in \mathrm{Fac}, 0 \le k < n!-1} h_\xi(k) \cdot p^k \\ &\le \frac{p-1}{p^{n!}} \cdot \sum_{m=0}^{n-1} p^{m!} \\ &\le \frac{p-1}{p^{n!}} \cdot n \cdot p^{(n-1)!} \\ &< \frac{(p-1) \cdot n}{p^n}, \end{aligned}$$

and again $h_\xi(x_n) \to 0$. □

Theorem 5.3. *Let $A = \mathbb{Z}$ or $A = \bigoplus_{k<\omega} \mathbb{Z}(p_k)$ ($p_k \in \mathbb{P}$, repetitions allowed), or $A = \mathbb{Z}(p^\infty)$ ($p \in \mathbb{P}$). Then there are a faithfully indexed sequence $(x_n)_{n<\omega}$ in A and a point-separating subgroup $H \subseteq \mathrm{Hom}(A, \mathbb{T})$ such that $|H| = \mathfrak{c}$ and $x_n \to 0$ in the space $(A, \mathcal{T}_H)$.*

Proof. We refer the reader for a detailed proof in the case $A = \mathbb{Z}$ to [29], [30]. When $A = \bigoplus_{k<\omega} \mathbb{Z}(p_k)$ or $A = \mathbb{Z}(p^\infty)$ there are, according to Theorem 5.1 or 5.2 respectively, a sequence (x_n) in A and a faithfully indexed family $\{h_\xi : \xi < \mathfrak{c}\} \subseteq \mathrm{Hom}(A, \mathbb{T})$ such that $h_\xi(x_n) \to 0$ for each $\xi < \mathfrak{c}$. It is then clear, as in [9], that with $H := \langle \{h_\xi : \xi < \mathfrak{c}\} \rangle \subseteq \mathrm{Hom}(A, \mathbb{T})$ we have $h(x_n) \to 0$ for each $h \in H$. □

Corollary 5.4. *Let G be an infinite Abelian group. There are a faithfully indexed sequence $(x_n)_{n<\omega}$ in G and a topology $\mathcal{T}_{H^*} \in \mathfrak{t}(G)$ such that $|H^*| = 2^{|G|}$ and $x_n \to 0$ in $(G, \mathcal{T}_{H^*})$.*

Proof. As indicated earlier, G contains algebraically a group A such that $A \simeq \mathbb{Z}$ or $A \simeq \bigoplus_{k<\omega} \mathbb{Z}(p_k)$ or $A \simeq \mathbb{Z}(p^\infty)$. Let $H \subseteq \mathrm{Hom}(A, \mathbb{T})$ and $(x_n)_n$ in A be as in Theorem 5.3, and set $H^* := \{k \in \mathrm{Hom}(G, \mathbb{T}) : k|A \in H\}$. Clearly $k(x_n) \to 0$ for each $k \in H^*$, so $x_n \to 0$ in $(G, \mathcal{T}_{H^*})$. If $|G| = \omega$ then $|H^*| = \mathfrak{c} = 2^\omega$ since $\mathfrak{c} = |H| \le |H^*| \le |\mathbb{T}^G| = \mathfrak{c}$. If $|G| > \omega$ we write

$$\mathbb{A}(\widehat{G}, A) := \{k \in \mathrm{Hom}(G, \mathbb{T}) : k \equiv 0 \text{ on } A\} \subseteq H^*;$$

then $|\mathbb{A}(\widehat{G}, A)| = 2^{|G/A|} = 2^{|G|}$ since algebraically $\mathbb{A}(\widehat{G}, A) = \mathrm{Hom}(G/A, \mathbb{T})$ and $|G| = |G/A|$, so $|H^*| = 2^{|G|}$ in this case also. □

We have arrived at the final result of this Section, which we view as a companion or "echo" to Theorem 4.1.

Theorem 5.5. *Every infinite Abelian group G admits a family $\mathcal{B}$ of totally bounded group topologies, with $|\mathcal{B}| = 2^{2^{|G|}}$, such that some (fixed) nontrivial*

sequence in G converges in each of the topologies in $\mathcal{B}$. One may arrange in addition that

(i) $w(G,\mathcal{T}) = 2^{|G|}$ *for each* $\mathcal{T} \in \mathcal{B}$, *and*

(ii) *for distinct* $\mathcal{T}_0$, $\mathcal{T}_1 \in \mathcal{B}$ *the spaces* $(G,\mathcal{T}_0)$ *and* $(G,\mathcal{T}_1)$ *are not homeomorphic.*

Proof. Let H^* be a subgroup of $\mathrm{Hom}(G,\mathbb{T})$ such that $|H^*| = 2^{|G|}$ and some nontrivial sequence $(x_n)_n$ in G satisfies $x_n \to 0$ in $(G,\mathcal{T}_{H^*})$. There is a subgroup H of H^* such that H separates points of G and $|H| = |G|$, and since $|H^*/H| = 2^{|G|}$ there is by Theorem 1.7 a faithfully indexed family $\{H_\xi : \xi < 2^{2^{|G|}}\}$ of (point-separating) groups such that $H \subseteq H_\xi \subseteq H^*$ for each ξ. Then $\mathcal{B} := \{\mathcal{T}_{H_\xi} : \xi < 2^{2^{|G|}}\}$ satisfies all requirements except perhaps (ii), and (ii) is handled as in the final sentences of the proof of Theorem 4.1. □

Remarks 5.6. (a) If $\mathcal{A}$ and $\mathcal{B}$ are as in Theorems 4.1 and 5.5, and if $\mathcal{T}_0, \mathcal{T}_1 \in \mathcal{A} \cup \mathcal{B}$ with $\mathcal{T}_0 \neq \mathcal{T}_1$, then the spaces $(G,\mathcal{T}_0)$ and $(G,\mathcal{T}_1)$ are not homeomorphic. For if both $\mathcal{T}_i \in \mathcal{A}$ or both $\mathcal{T}_i \in \mathcal{B}$ this is already proved, while if (say) $\mathcal{T}_0 \in \mathcal{A}$ and $\mathcal{T}_1 \in \mathcal{B}$ then $(G,\mathcal{T}_1)$ has a nontrivial convergent sequence and $(G,\mathcal{T}_0)$ does not.

(b) We emphasize that for an infinite Abelian group G, the algebraic structure of a point-separating subgroup $H \subseteq \mathrm{Hom}(G,\mathbb{T})$ by no means determines the topology $\mathcal{T}_H$ on G. It is noted explicitly in [29], [30] that when $G = \mathbb{Z}$ then every one of the topologies in the families $\mathcal{A}$ and $\mathcal{B}$ (as in Theorems 4.1 and 5.5) can be chosen of the form $\mathcal{T}_H$ with $H \subseteq \mathbb{T} = \mathrm{Hom}(G,\mathbb{T})$ and with $H \simeq \bigoplus_{\xi<\mathfrak{c}} \mathbb{Z}_\xi$.

6. Concluding Remarks

It was shown in [1], assuming MA, that there is a measurable subgroup H of $\mathbb{T}$ such that $\lambda(H) = 0$ and no nontrivial sequence converges in the space $(\mathbb{Z},\mathcal{T}_H)$. Responding to a question in [1], the authors of [19] achieved the same result in ZFC. It is natural to ask if the comparable statement holds for each infinite Abelian group. In detail:

Question 6.1. Does every infinite Abelian group G admit a group topology $\mathcal{T} = \mathcal{T}_H \in \mathfrak{t}(G)$ with no nontrivial convergent sequences such that H is measurable in the compact group $\mathrm{Hom}(G,\mathbb{T}) = \widehat{G_d}$ and $\lambda(H) = 0$?

Our convergent sequences in Theorems 5.1 and 5.2 were quite “thin”. Viewing those results from another perspective, a natural question arises.

Question 6.2. Let G be an infinite Abelian group. For which faithfully indexed sequences $(x_n)_n$ in G is there a topology $\mathcal{T}_H \in \mathfrak{t}(G)$ with $w(G,\mathcal{T}_H) = |H| = 2^{|G|}$ such that $x_n \to 0$ in $(G,\mathcal{T}_H)$?

Many questions arise in the non-Abelian context. Perhaps this one of Saeki and Stromberg [31] bears repeating.

Question 6.3. [31]. Does every infinite (not necessarily Abelian) compact group have a dense, nonmeasurable subgroup?

Remark 6.4. Malykhin and Shapiro [27] showed by a direct argument that for every faithfully indexed sequence $x = (x_n)_n$ in an Abelian group G there is $h_x \in \mathrm{Hom}(G, \mathbb{T})$ such that $h_x(x_n) \not\to 0$. Thus every Abelian G with $|G| = \alpha$ admits a topology $\mathcal{T}_H \in \mathfrak{t}(G)$ such that $|H| = w(G, \mathcal{T}_H) = \alpha^\omega$ and no nontrivial sequence converges in $(G, \mathcal{T}_H)$. This statement can be improved slightly using Theorem 1.6 above and this result from [5](2.2): Every compact group K with $w(K) = \alpha$ contains a dense, countably compact, subgroup H such that $|H| = (\log\,\alpha)^\omega$. Since such H is pseudocompact and hence nonmeasurable, we have (beginning with Abelian G such that $|G| = \alpha$ and taking $K := \mathrm{Hom}(G, \mathbb{T}) = \widehat{G_d}$) that $w(G, \mathcal{T}_H) = (\log\,\alpha)^\omega$ and $(G, \mathcal{T}_H)$ has no convergent sequences. (We note in this connection that $(\log\,\alpha)^\omega < 2^\alpha$ for every strong limit cardinal α such that $\mathrm{cf}(\alpha) > \omega$.) This discussion suggests a question.

Question 6.5. Given an infinite Abelian group G, what is the minimal weight of a topology in $\mathcal{T} \in \mathfrak{t}(G)$ such that no nontrivial sequence converges in $(G, \mathcal{T})$? For which G is this $2^{|G|}$?

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